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Stress distribution along the contour of a circular opening in wooden plate loaded by in-plane bending moment

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Abstract: *Stress distribution along the contour of a circular opening in wooden plate loaded by in-plane bending moment.* The analytical solution of stresses around the circular hole boundary in two-dimensional wooden plate modelled as orthotropic linear elastic material is presented here. The orthotropic plate with the circular hole is subjected to an in-plane bending loading. The aim is to know the influence of the circular hole and of the principal directions of elasticity on concentration of stresses. Realised computations of the hoop stress on the opening boundary and of the stress concentration factor are based on the linear theory of anisotropic bodies with using of a complex variable method.

Keywords: stress concentration, circular hole, wooden plate, in-plane bending.

INTRODUCTION

It is well known that holes cause serious problems of stress concentrations due to the geometry discontinuity. These problems are even more serious in structures of materials with anisotropic behaviour. In order to predict the structural behaviour of these structures, it is necessary to study the effect of anisotropy and type of loading on stress distribution around the holes.

Analytical calculation of stress concentrations in plate with infinite dimensions under mechanical loads have been performed by many authors mainly using the methods of complex valued stress functions and conformal mappings. Ukadgaonker and Rao [5] adapted Savin's formulation to get general solution for in plane loading problem. Mathematically elegant and technically powerful in determining the two-dimensional deformations of anisotropic elastic solids is Stroh formalism [3], [4], [1].

Here the Lechnickij's [2] complex variable approach is used for calculation of stress concentration around circular opening in wooden plate modelled as orthotropic material.

THEORETICAL BACKGROUND

To get solution of two dimensional anisotropic elasticity problems, it is necessary to solve basic equation of theory of the plane elasticity of anisotropic bodies (1) for given boundary conditions

$$a_{22}\frac{\partial^4 U}{\partial x^4} - 2a_{26}\frac{\partial^4 U}{\partial x^3 \partial y} + (2a_{12} + a_{66})\frac{\partial^4 U}{\partial x^2 \partial y^2} - 2a_{16}\frac{\partial^4 U}{\partial x \partial y^3} + a_{11}\frac{\partial^4 U}{\partial y^4} = 0. \tag{1}$$

The Lechnickij's formalism the given problems formulate in terms of the analytic functions, $\Phi_k(z_k)$, of the complex variable, $z_k = x_k + iy_k \quad (k = 1,\ 2)$, $i = \sqrt{-1}$, where

$x_k = x + \alpha_k y$, $y_k = \beta_k y$, $(k = 1,\ 2)$.

The parameters α_k and β_k are the real and imaginary parts of complex parameters

$$\mu_k = \alpha_k + i\beta_k \, , \tag{2}$$

which can be determined from the following characteristic equation

$$a_{11}\mu^4 - 2a_{16}\mu^3 + (2a_{12} + a_{66})\mu^2 - 2a_{26}\mu + a_{22} = 0 \, , \tag{3}$$

where the roots μ_k are always complex or purely imaginary in conjugate pairs as $\mu_1, \ \overline{\mu}_1$; $\mu_2, \ \overline{\mu}_2$. As it is evident from Eq. (3), these complex parameters represented by Eq. (2) depend on the compliance coefficients a_{ij} (i,j = 1, 2, 6) of the anisotropic plate. The Airy′s stress function $U(x,y)$ can be represented by arbitrary functions F_k of variables z_k

$$U(x,y) = F_1(z_1) + F_2(z_2) + \overline{F_1(z_1)} + \overline{F_2(z_2)}. \tag{4}$$

By introducing the designations

$$\Phi_k(z_k) = \frac{\mathrm{d}F_k}{\mathrm{d}z_k} \quad (k = 1,\ 2) \, , \qquad \Phi_k'(z_k) = \frac{\mathrm{d}\Phi_k}{\mathrm{d}z_k} \quad (k = 1,\ 2) \, ,$$

the stress components in terms of $\Phi_k(z_k)$) can be formulated as

$$\begin{aligned} \sigma_x &= 2\,\mathrm{Re}\left[\mu_1^2 \Phi_1'(z_1) + \mu_2^2 \Phi_2'(z_2)\right] \, , \\ \sigma_y &= 2\,\mathrm{Re}\left[\Phi_1'(z_1) + \Phi_2'(z_2)\right] \, , \\ \tau_{xy} &= -2\,\mathrm{Re}\left[\mu_1 \Phi_1'(z_1) + \mu_2 \Phi_2'(z_2)\right] \, . \end{aligned} \tag{5}$$

Substituting the expressions for σ_x , σ_y , τ_{xy} to transformation formulas for normal and tangential stress components in a plane with an arbitrary directed normal n, these stress components are

$$\begin{aligned} \sigma_n &= 2\,\mathrm{Re}\left\{\left[\cos(n,y) - \mu_1 \cos(n,x)\right]^2 \Phi_1'(z_1) + \left[\cos(n,y) - \mu_2 \cos(n,x)\right]^2 \Phi_2'(z_2)\right\} \, , \\ \tau_n &= 2\,\mathrm{Re}\left\{\left[\cos(n,y) - \mu_1 \cos(n,x)\right] \times \left[\cos(n,x) + \mu_1 \cos(n,y)\right] \Phi_1'(z_1) + \right. \\ &\quad \left. + \left[\cos(n,y) - \mu_2 \cos(n,x)\right] \times \left[\cos(n,x) + \mu_2 \cos(n,y)\right] \Phi_2'(z_2)\right\} \, . \end{aligned} \tag{6}$$

CONCENTRATION OF STRESSES IN ORTHOTROPIC PLATE WITH CIRCULAR OPENING AT BENDING BY IN-PLANE MOMENTS

A rectangular beam, which is a plate with a circular opening at the center, is bent by moments M applied to both sides in the middle plane. The principal directions of elasticity, parallel with axes x and y, do not coincide with the direction of plate sides. Their orientation is characterized by angle φ (Fig. 1).

The largest stress concentration is on the opening edge. The normal stress σ_θ acting on areas normal to opening edge, i.e., on radial planes located at the edge of the opening, is one principal stress, second principal stress, as it is evident from given boundary condition, is equal zero. The stresses in the plate are obtained by summing stresses in a solid beam plate subjected to pure bending and the stresses obtained by functions $\Phi_k(z_k)$. So the hoop stress σ_θ at individual contour points given by angle θ is expressed by formula

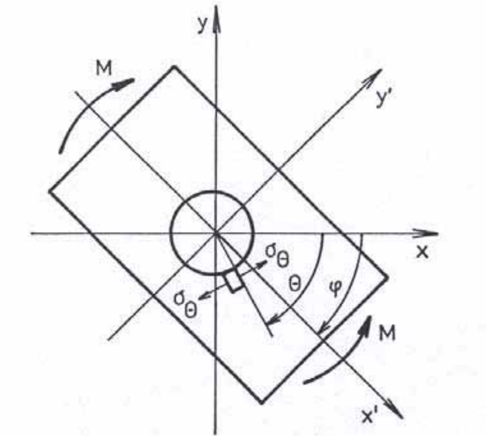


Fig. 1 Problem configuration

$$\sigma_\theta = \frac{Ma}{2J} \cdot \frac{E_\theta}{E_1} \Big(k\left[1 - k - (1 + k + n)\cos 2\theta\right]\sin^3\varphi\cos\theta + \\ \left\{ n^2 + k(k + 2n - 1) + \left[n(1 + n) + k(1 + k + 2n)\right]\cos 2\theta \right\} \times \\ \times \sin^2\varphi\cos\varphi\sin\theta - \left[(1 + n)^2 - k - (k + n + 1)(1 + n)\right]\cos 2\theta \times \\ \times \sin\varphi\cos^2\varphi\cos\theta + \left[1 - k - (1 + k + n)\cos 2\theta\right]\cos^3\varphi\sin\theta \Big) , \tag{7}$$

where J is the inertia moment of the transverse cross section of a solid (of the unweakened plate), E_1 is Young's modulus in principal direction, E_θ is Young's modulus in the direction tangent to the opening contour, a is a radius of the opening, k and n denote

$$k = -\mu_1\mu_2 , \qquad n = -i(\mu_1 + \mu_2),$$

θ is the polar angle measured from the x-axis and φ is an angle between the principal direction of elasticity and plate axis.

RESULTS OF NUMERICAL EXAMPLES

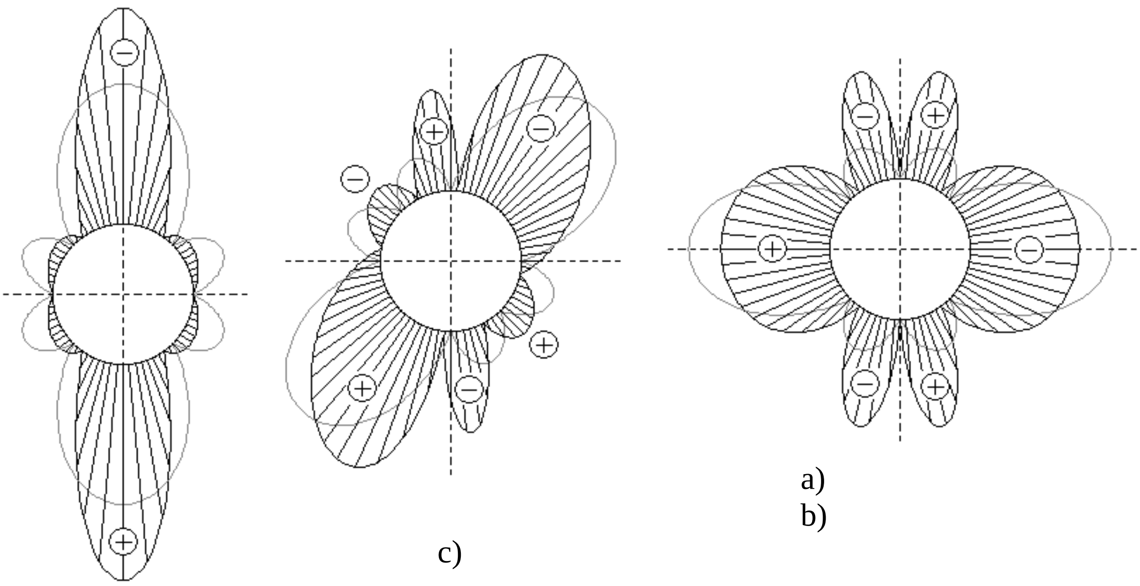
Calculation of stress distribution was made for tangential wooden plate of *Picea Excelsa* modeled as the orthotropic plate with the circular opening. The rectangular plate is bent by moments *M*, radius *a* of the opening is taken to be small in comparison with the length of the plate sides and the principal directions of elasticity are assumed as axes *x* and *y*. The axes *x* (parallel with wood fibers) forms an angle φ with the plate axis x' .

Used engineering constants in principal material directions - Young's moduli E_1, E_2 , Poisson ratios ν_1 , ν_2 and shear modulus G_{12} - have values:

$E_1 =$ 9 290 MPa , $E_2 =$ 650 MPa , $\nu_1 = 0.420$, $\nu_{12} = 0.033$, $G_{12} =$ 870 MPa .

Complex parameters calculated from equation (2) are: $\mu_1 = 0 + 1.331i$, $\mu_2 = 0 + 2.840i$.

Distribution of the hoop stress σ_θ on the hole boundary for solved cases of the angle φ (φ= 0°, φ= 45° and φ= 90°) is presented in Fig. 2, where a ash-grey line displays the hoop



a)
b)
c)

Fig. 2 Distribution of σ_θ : a) φ= 0°, b) φ= 45°, c) φ= 90°

stress σ_θ distribution in isotropic material. To denote the highest stress caused by the hole, the stress concentration factor (SCF) is used. The stress concentration factor for this case of loading is defined to be the maximum stress σ_θ at the opening boundary divided by a stress

$$\sigma = \frac{Ma}{J}$$ from the unweakened plate.

Calculated stress concentration factors of wooden plate for observed orientation of fibers and SCF in the isotropic material at given manner of loading are given in Table 1.

Table 1. Stress concentration factors

	Picea Excelsa			Isotropic material
	$\varphi = 0°$	$\varphi = 45°$	$\varphi = 90°$	
SCF	3,09	2,26	1,59	2,00

CONCLUSION

Circular opening causes stress concentration in the wooden plate loaded by in-plane bending moments. Distribution of the hoop stress at the opening boundary depends on angle φ between the plate axis and fibers direction. The stress concentration factor achieves maximal value when fibers orientation is parallel with plate axis. In this case the maximal hoop stress is at contour point given by angle $\theta = 90°$ and 270°. When φ= 45°, extreme value of hoop stress is in locations about $\theta = 118°$ and 298°. For φ= 90°, these locations are about $\theta = 77°$

and 257°. Extreme values σ_θ in isotropic material are in regions, where the opening edge crosses the plate axis x' .

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Streszczenie: *Rozkład naprężeń wokół okrągłego otworu w drewnianej płycie obciążonej momentem zginającym*. Przedstawiono rozwiązanie analityczne rozkładu naprężeń wokół otworu w dwuwymiarowej płycie drewnianej modelowanej jako ortotropowy materiał elastyczny. Ortotropowa płyta poddana jest naprężeniom zginającym. Celem pracy było określenie wpływu otworów i głównych kierunków sprężystości na rozkład naprężeń. Wyliczenia naprężeń obwodowych na granicy otworu i wpływu roskłądu naprężeń oparto na teorii liniowej ciał anizotropowych używając metody wielu zmiennych.

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